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Necessary conditions of convergence of Hermite–Fejér interpolation polynomials for exponential weights

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Abstract

This paper gives the conditions necessary for weighted convergence of Hermite–Fejér interpolation for a general class of even weights which are of exponential decay on the real line or at the end points of $(-1, 1)$. The results of this paper guarantee that the conditions of Theorem 2.3 in [11] are optimal.

1. Introduction

For a function $f : (a, b) \to \mathbb{R},\ -\infty \leqslant a < b \leqslant \infty$ and a set

$$\chi_n := \{x_{1n}, x_{2n}, \ldots, x_{nn}\},\ n \geqslant 1$$

of pairwise distinct nodes let $H_n[\chi_n; f]$ denote the Hermite–Fejér interpolation polynomials of degree $\leqslant 2n-1$ to f with respect to χ_n. In fact, $H_n[\chi_n; f]$ is the unique polynomial of degree $\leqslant 2n-1$ satisfying

$$H_n[\chi_n; f](x_{jn}) = f(x_{jn}) \quad \text{and} \quad H_n'[\chi_n; f](x_{jn}) = 0 \tag{1.1}$$

for $j = 1, 2, \ldots, n$.

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This paper deals with Hermite–Fejér interpolations with respect to χ_n whose elements are the zeros of a sequence of orthogonal polynomials. More precisely, in this paper we consider $w_Q(x) := \exp(-Q(x))$, where $Q : I \to \mathbb{R}$ is even, continuous, and of at least polynomial growth at the end of interval I and I is either $(-1, 1)$ or $\mathbb{R}$. Then χ_n consists of the zeros $\{x_{j,n}(w_Q^2)\}_{j=1}^n$ of the n-th orthonormal polynomial $p_n(w_Q^2, x)$,

$$p_n(x) := p_n(w_Q^2, x) = \gamma_n(w_Q^2)x^n + \textit{lower degree terms } (\gamma_n(w_Q^2) > 0)$$

with respect to w_Q^2, defined by the condition

$$\int_I p_n p_m w_Q^2(x)\,dx = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots.$$

Then all $\{x_{j,n}(w_Q^2)\}_{j=1}^n$ belongs to I, which we arrange as

$$x_{n,n}(w_Q^2) < x_{n-1,n}(w_Q^2) < \cdots < x_{2,n}(w_Q^2) < x_{1,n}(w_Q^2).$$

Let $H_n[w_Q^2; \cdot]$ be the Hermite–Fejér interpolation operator with respect to the zeros $\{x_{j,n}(w_Q^2)\}_{j=1}^n$ of $p_n(w_Q^2; x)$.

Our main concern is the following problem: *What is a necessary and sufficient condition on $u(x)$ and $w(x)$ that the relation*

$$\lim_{n\to\infty} \left\| \left(f - H_n[w_Q^2; f]\right) w \right\|_{L_\infty(I)} = 0$$

holds for every continuous function satisfying $\lim_{|x|\to\infty \text{ or } 1} \left| f(x)w_Q^2(x)u(x) \right| = 0$?

Several sufficient conditions for weighted convergence of Hermite–Fejér interpolation polynomials are obtained. See [6,11,12,16,25,31] and the references therein. In particular, [11,16,31] gave sufficient conditions of our problem with respect to the weights decaying exponentially at the end points. There is a vast literature dealing with necessary and sufficient conditions for weighted convergence of Lagrange interpolation for even Freud, Erdős, and exponential weights on $(-1, 1)$. We refer the reader to [1–5,7–9,17,19–22,24,26,28–31] and the many references cited therein. Especially, some necessary conditions for weighted convergence of Lagrange interpolation with respect to these weights were given in [4,8,9,28,30]. In this paper we intend to give the conditions necessary for weighted convergence of Hermite–Fejér interpolation polynomials with respect to the weights decaying exponentially at the end points.

This paper is organized as follows: in Section 2, we introduce our admissible class of weights and state the main results. In Section 3, we present some lemmas and prove the results of Section 2. In Section 4, we especially apply our main theorems to Freud and Erdős weights cases. Finally, in Section 5, we recall some notations, bounds on orthogonal polynomials and related estimates.

2. Main results

We first introduce some notations which we use in the following. For any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of nonzero real numbers(or functions), we write $b_n \lesssim c_n$, if there exists a constant $C > 0$, independent of n(and x) such that $b_n \leqslant C c_n$ for n large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote by $\mathcal{P}_n$ the space of polynomials of degree at most n. Let $I+$ be either $(0, \infty)$ if $I = \mathbb{R}$ or $(0, 1)$ if $I = (-1, 1)$.

We now introduce an admissible class of weights.

Definition 2.1. Let $w_Q(x) = \exp(-Q(x))$ where $Q(x) : I \to \mathbb{R}$ is even, continuous, and

(a) $Q''(x)$ is continuous in $I+$ and $Q''(x), Q'(x) \geqslant 0$ in $I+$,

(b) the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \; x \in I+ \tag{2.1}$$

satisfies for large enough x or x close enough to 1

$$T(x) \sim \frac{xQ'(x)}{Q(x)}. \tag{2.2}$$

Moreover T, satisfies either:

(b1) There exist $A > 1$ and $B > 1$ such that

$$A \leqslant T(x) \leqslant B, \; x \in I+. \tag{2.3}$$

(b2) T is increasing in $I+$ with $\lim_{x\to 0+} T(x) > 1$. If $I = \mathbb{R}$,

$$\lim_{|x|\to\infty} T(x) = \infty$$

and if $I = (-1, 1)$, for x close enough to ± 1 and some $A > 2$,

$$T(x) \geqslant \frac{A}{1 - x^2}.$$

Then, $w_Q(x)$ is called an admissible weight and we write $w_Q \in \mathcal{A}$.

We call $w_Q(x)$ a Freud weight in the case of (b1). In the case of (b2), we call it an Erdős weight in case $I = \mathbb{R}$ or an exponential weight on $(-1, 1)$ in case $I = (-1, 1)$. Freud weights are characterized by smooth polynomial decay of $Q(x)$ at infinity and Erdős weights by their faster than smooth polynomial decay at infinity. Exponential weights on $(-1, 1)$ decay strongly near ± 1 as exponentials decay faster than classical Jacobi weights. They violate the well-known Szegő condition for orthogonal polynomials (cf. [10, p. 208]).

The author gave a sufficient condition for our problem in [11]. In the following, we state the extended Szabados' result of [11].

Theorem 2.2 (*Jung [11]*). *Let $w_Q \in \mathcal{A}$, $u(x) := |Q'(x)|$, and $v(x) := (|x| + 1)^{-1/3}$. For a continuous function f on I with*

$$\lim_{|x|\to\infty \text{ or } 1} \left| f(x) w_Q^2(x) u(x) \right| = 0,$$

it holds that

$$\lim_{n\to\infty} \left\| \left(f - H_n[w_Q^2; f]\right) w_Q^2 v \right\|_{L_\infty(I)} = 0.$$

In the following, a necessary condition for the extended Szabados' result of [11] is given.

Theorem 2.3. *Let $w_Q \in \mathcal{A}$. Suppose $v : I \to \mathbb{R}^+$ is a measurable function satisfying that*

$$\lim_{x\to\infty \text{ or } 1} x v(x) Q^{-2/3}(x)/\log(1+|x|) = \infty.$$

Then there exists a continuous function $f : I \to \mathbb{R}$ satisfying that

$$\lim_{|x|\to\infty \text{ or } 1} |f(x) w_Q^2(x) Q'(x) \log(1+|x|)| = 0$$

such that

$$\limsup_{n\to\infty} \|H_n[w_Q^2; f] w_Q^2 v\|_{L_\infty(I)} = \infty.$$

Theorem 2.4. *Let $w_Q \in \mathcal{A}$. If for every continuous function f defined on I satisfying that*

$$\lim_{|x|\to\infty \text{ or } 1} |f(x) w_Q^2(x) Q'(x) \log(1+|x|)| = 0 \tag{2.4}$$

it holds that

$$\lim_{n\to\infty} \left\| (f - H_n[w_Q^2; f]) w_Q^2 v \right\|_{L_\infty(I)} = 0$$

then it is necessary that

$$\lim_{x\to\infty \text{ or } 1} x v(x) Q^{-2/3}(x)/\log(1+|x|) < \infty. \tag{2.5}$$

Remark 2.5. Let $w_Q(x) := \exp(-|x|^a/2)$ with $a > 1$, and $v(x) := (|x|+1)^{-\Delta}$. Then the necessary condition for Δ is $\Delta \geqslant 1 - (2a)/3$, because for large $|x| > 0$ by Theorem 2.4

$$v(x) x Q^{-2/3}(x)/\log(1+|x|) \sim |x|^{1-\Delta-(2a)/3}/\log(1+|x|) < \infty$$

should hold. Therefore, the condition $\Delta \geqslant 1/3$ is necessary in order that the weighted Hermite–Fejér interpolation polynomials converge for all $w_Q(x) := \exp(-|x|^a/2)$ with $a > 1$ and the factor $(1+|x|)^{-1/3}$ in Theorem 2.2 is optimal.

Moreover, we give a necessary condition for L_p convergence in the following. Let a_u for $u > 0$ be the Mhaskar–Rakhamanov–Saff number. See Section 5, Appendix.

Theorem 2.6. *Let $w_Q \in \mathcal{A}$ and $0 < p \leqslant \infty$ be given. Let $w \geqslant 0$ and u be an even and non decreasing function defined on I. Suppose for some fixed $0 < \delta < 1$ and $\sigma > 1$,*

$$\limsup_{n\to\infty} u^{-1}(a_{\delta n}) T^{-3/2}(a_n) a_n^{1-1/p} \times \left\| p_n^2(x) w(x) \right\|_{L_p(|x|\leqslant a_{\sigma n})} / \log^{1+1/p} a_n = \infty. \tag{2.6}$$

Then there exists a continuous function $g : I \to \mathbb{R}$ *satisfying that*

$$\left\| g(x) w_Q^2(x) u(x) \right\|_{L_p(I)} < \infty \tag{2.7}$$

such that

$$\limsup_{n\to\infty} \|H_n[w_Q^2; g](x) w(x)\|_{L_p(I)} = \infty.$$

Theorem 2.7. *Let* $w_Q \in \mathcal{A}$ *and* $0 < p \leqslant \infty$ *be given. Let* $w \geqslant 0$ *and u be an even and non decreasing function defined on I. If for every continuous function g defined on I satisfying* (2.7) *it holds that*

$$\lim_{n\to\infty} \left\| (g - H_n[w_Q^2; g]) w \right\|_{L_p(I)} = 0$$

then it is necessary that for some fixed $0 < \delta < 1$ *and* $\sigma > 1$,

$$\limsup_{n\to\infty} u^{-1}(a_{\delta n}) T^{-3/2}(a_n) a_n^{1-1/p} \left\| p_n^2(x) w(x) \right\|_{L_p(|x| \leqslant a_{\sigma n})} / \log^{1+1/p} a_n < \infty. \tag{2.8}$$

3. Lemmas and proofs

To prove the theorems, we need some lemmas. In the following, constants independent of n and x are denoted by $C, C_1, C_2, \ldots$. The symbol C does not necessarily denote the same constant in different occurrences.

By (1.1), we have (cf. [32, p. 330])

$$H_n[w_Q^2; f](x) := \sum_{k=1}^{n} \left(1 - \frac{p_n{}''(x_{kn})}{p_n'(x_{kn})} (x - x_{kn}) \right) f(x_{kn}) l_{kn}^2(x), \tag{3.1}$$

where $l_{kn}(x)$ is the fundamental Lagrange interpolation polynomial ([10, p. 23]), given by

$$l_{kn}(w_Q^2; x) := \frac{p_n(w_Q^2; x)}{p_n'(w_Q^2; x_{kn})(x - x_{kn})}, \quad k = 1, 2, \ldots, n.$$

Define

$$\bar{Q}(x,t) := \frac{Q'(t) - Q'(x)}{t - x}, \quad x, t \in I \setminus \{0\}, \tag{3.2}$$

$$\rho_n := \rho_n(w_Q^2) := \gamma_{n-1}(w_Q^2)/\gamma_n(w_Q^2),$$

and

$$A_n(x) := 2\rho_n \int_I (p_n w_Q)^2(t) \bar{Q}(x,t)\, dt. \tag{3.3}$$

Furthermore, define the Christoffel numbers

$$\lambda_{kn} := \int_I l_{kn}^2(x) w_Q^2(x)\, dx = \int_I l_{kn}(x) w_Q^2(x)\, dx \quad k = 1, 2, \ldots, n$$

and δ_n, $\Psi_n(x)$ and their properties are shown in Section 5, Appendix.

Lemma 3.1. *Let $w_Q \in \mathcal{A}$. Then for $1 \leqslant k \leqslant n$*

$$l_{kn}(x) = \lambda_{kn}\rho_n p_{n-1}(x_{kn}) \frac{p_n(x_{kn})}{x - x_{kn}}; \tag{3.4}$$

$$\lambda_{kn}\rho_n p_{n-1}^2(x_{kn}) = \frac{1}{A_n(x_{kn})}; \tag{3.5}$$

$$A_n'(x_{kn}) = 2\rho_n \int_I (p_n w_Q)^2(t) \frac{\bar{Q}(x,t)}{t - x_{kn}}\, dt \tag{3.6}$$

and

$$\frac{p_n''(x_{kn})}{p_n'(x_{kn})} = 2Q'(x_{kn}) + \frac{A_n'(x_{kn})}{A_n(x_{kn})}. \tag{3.7}$$

Proof. (3.4) is in [10, p. 23–34]; (3.5) is in [25, (5.9)]; (3.6) is in [25, p. 579, in the proof of Lemma 5.3]; and (3.7) is in [25, (5.5)]. □

Lemma 3.2. *Let $w_Q \in \mathcal{A}$. Then there exist constants $C_1, C_2 > 0$ such that uniformly for $1 \leqslant k \leqslant n$ and n,*

$$Q'(x_{kn}) - C_1 \leqslant \frac{p_n''(x_{kn})}{p_n'(x_{kn})} \leqslant 3Q'(x_{kn}) + C_2.$$

Proof. We will follow the method of Lemma 5.3 in [25]. From (3.6), we can obtain

$$\begin{aligned} \left|A_n'(x_{kn})\right| &\leqslant 2\rho_n \int_{|t-x_{kn}|\geqslant 2(1+Q'(x_{kn}))^{-1}} (p_n w_Q)^2(t) \frac{|\bar{Q}(x_{kn},t)|}{|t-x_{kn}|}\, dt \\ &\quad + 2\rho_n \int_{|t-x_{kn}|< 2(1+Q'(x_{kn}))^{-1}} (p_n w_Q)^2(t) \frac{|\bar{Q}(x_{kn},t)|}{|t-x_{kn}|}\, dt \\ &:= I_1 + I_2. \end{aligned}$$

Since $\bar{Q}(x,t) \geqslant 0$ from (a) of Definition 2.1, by (3.3) we have

$$\begin{aligned} I_1 &\leqslant 2\rho_n \int_I \frac{1}{2}(1 + Q'(x_{kn}))(p_n w_Q)^2(t)|\bar{Q}(x_{kn},t)|\, dt \\ &= \tfrac{1}{2}(1 + Q'(x_{kn}))A_n(x_{kn}). \end{aligned}$$

Next, we estimate I_2. Suppose that $x_{kn} > 0$ and let

$$u := \frac{2}{\pi}\int_0^1 x_{kn} t Q'(x_{kn}t)/\sqrt{1-t^2}\, dt.$$

Then $a_u = x_{kn}$. See Section 5, Appendix. Then since $|t| \leqslant a_u + 2(1+Q'(a_u))^{-1}$, by (A.5) we have

$$\left|\frac{t}{x_{kn}}\right| \leqslant 1 + \frac{2}{a_u Q'(a_u)} \leqslant 1 + \frac{C}{uT^{1/2}(a_u)}.$$

Here, since $C\frac{T^{1/2}(a_u)}{u} = o(1)$ as $u \to \infty$ by (A.8), there exists a constant $c > 1$ such that by (A.7),

$$|t| \leqslant \left(1 + \frac{C}{uT^{1/2}(a_u)}\right) a_u \leqslant \left(1 + \frac{o(1)}{T(a_u)}\right) a_u \leqslant a_{cu}, \quad \text{as } u \to \infty.$$

By the mean value property, there exists τ between x_{kn} and t such that if $0 < |\tau| \leqslant 1/2$ then

$$\left|Q'(t) - Q'(x_{kn})\right| = Q''(|\tau|)|t - x_{kn}| \leqslant C_1 \frac{2}{1 + Q'(x_{kn})} \leqslant C_2$$

and if $|\tau| > 1/2$ then

$$\begin{aligned}
\left|Q'(t) - Q'(x_{kn})\right| &= Q''(|\tau|)|t - x_{kn}| = \frac{T(|\tau|) - 1}{|\tau|} Q'(|\tau|)|t - x_{kn}| \ \text{ by (2.1)} \\
&\leqslant \frac{4T(|\tau|)}{1 + Q'(x_{kn})} Q'(|\tau|) \leqslant \frac{4T(|\tau|)}{Q'(a_u)} Q'(|\tau|) \\
&\leqslant C_1 \frac{T(a_{cu})}{Q'(a_u)} \frac{Q'(a_{cu})}{Q'(a_u)} Q'(a_u) \ (\because 0 < |\tau| \leqslant \max\{a_u, |t|\} \leqslant a_{cu}) \\
&\leqslant C_2 \frac{T(a_u)}{Q'(a_u)} Q'(a_u) \ \text{ by (A.6)} \\
&\leqslant C_3 \frac{a_u T^{1/2}(a_u)}{u} Q'(a_u) \ \text{ by (5.5) with } j = 1 \\
&\leqslant C_4 \left(\frac{a_u}{u}\right)^{\varepsilon} Q'(a_u) \quad \text{for some } \varepsilon > 0 \quad \text{by (A.8)} \\
&\leqslant \frac{1}{4}(Q'(x_{kn}) + C_5) \\
&\quad \left(\because \lim_{u\to\infty} \frac{a_u}{u} = 0 \text{ by (4.1) or (b2) conditions}\right).
\end{aligned}$$

Therefore, since for I_2

$$\left|Q'(t) - Q'(x_{kn})\right| = Q''(|\tau|)|t - x_{kn}| \leqslant \frac{1}{4}(Q'(x_{kn}) + C),$$

we have by (3.2), (3.4), and (3.5)

$$\begin{aligned}
I_2 &\leqslant 2\rho_n \int_{|t-x_{kn}|<2(1+Q'(x_{kn}))^{-1}} (p_n w_Q)^2(t) \frac{|\bar{Q}(x_{kn}, t)|}{|t - x_{kn}|}\, dt \\
&= 2\rho_n \int_{|t-x_{kn}|<2(1+Q'(x_{kn}))^{-1}} \left(\frac{p_n w_Q(t)}{|t - x_{kn}|}\right)^2 \left|Q'(t) - Q'(x_{kn})\right|\, dt \\
&\leqslant \frac{1}{2}\left(Q'(x_{kn}) + C\right) \rho_n \int_{|t-x_{kn}|<2(1+Q'(x_{kn}))^{-1}} \left(\frac{p_n w_Q(t)}{|t - x_{kn}|}\right)^2 dt
\end{aligned}$$

$$\leqslant \frac{1}{2}\left(Q'(x_{kn})+C\right)\rho_n \int_I \left(\frac{p_n w_Q(t)}{|t-x_{kn}|}\right)^2 dt$$
$$= \frac{1}{2}\left(Q'(x_{kn})+C\right)\rho_n \left(\lambda_{kn}\rho_n p_{n-1}(x_{kn})\right)^{-2} \int_I \left(l_{kn} w_Q(t)\right)^2 dt$$
$$\leqslant \frac{1}{2}\left(Q'(x_{kn})+C\right) A_n(x_{kn}).$$

From the estimations for I_1 and I_2, we have

$$\left|A_n'(x_{kn})/A_n(x_{kn})\right| \leqslant Q'(x_{kn})+C.$$

Therefore, there exist constants C_1 and $C_2 > 0$ such that we obtain by (3.7)

$$Q'(x_{kn})-C_1 \leqslant \frac{p_n{}''(x_{kn})}{p_n'(x_{kn})} \leqslant 3Q'(x_{kn})+C_2. \quad \Box$$

Lemma 3.3. *There exists a constant $C \in I+$ such that for $x_{kn} < -C$ and $x > 0$*

$$\frac{1}{2}(x-x_{kn})Q'(x_{kn}) \leqslant \frac{p_n{}''(x_{kn})}{p_n'(x_{kn})}(x-x_{kn})-1 \leqslant 4Q'(x_{kn})(x-x_{kn}).$$

Proof. Since $\lim_{|x|\to\infty \text{ or } 1} Q'(x)=\infty$, there exists a constant $C>0$ such that for $x_{kn} < -C$ and $x>0$,

$$\begin{aligned}
\frac{p_n{}''(x_{kn})}{p_n'(x_{kn})}(x-x_{kn})-1 &\geqslant \left(Q'(x_{kn})-C_1\right)(x-x_{kn})-1\\
&= (x-x_{kn})\left(Q'(x_{kn})-C_1-\frac{1}{x-x_{kn}}\right)\\
&= (x-x_{kn})\left(\frac{1}{2}Q'(x_{kn})+\frac{1}{2}Q'(x_{kn})-C_1-\frac{1}{x-x_{kn}}\right)\\
&\geqslant (x-x_{kn})\left(\frac{1}{2}Q'(x_{kn})+\frac{1}{2}Q'(x_{kn})-C_1-\frac{1}{|x_{kn}|}\right)\\
&\geqslant \frac{1}{2}(x-x_{kn})Q'(x_{kn})
\end{aligned}$$

and

$$\begin{aligned}
\frac{p_n{}''(x_{kn})}{p_n'(x_{kn})}(x-x_{kn})-1 &\leqslant \left(3Q'(x_{kn})+C_2\right)(x-x_{kn})-1\\
&\leqslant (x-x_{kn})\left(3Q'(x_{kn})+C_2\right)\\
&\leqslant 4(x-x_{kn})Q'(x_{kn})
\end{aligned}$$

where C_1 and C_2 are the same as in Lemma 3.2. $\Box$

Now, let $K>0$ be a constant satisfying the conditions of Lemma 3.3, i.e. for $x_{kn} < -K$ and $x>0$

$$\frac{1}{2}(x-x_{kn})Q'(x_{kn}) \leqslant \frac{p_n{}''(x_{kn})}{p_n'(x_{kn})}(x-x_{kn})-1 \leqslant 4Q'(x_{kn})(x-x_{kn}). \tag{3.8}$$

Proof of Theorem 2.3. Let $0 < K < J$ on $I+$. Then define a continuous function f satisfying $f(x)w_Q^2(x)Q'(x)\log(1+|x|) = -1$ on $x < -J$, $0 \leqslant -f(x)w_Q^2(x)Q'(x)\log(1+|x|) \leqslant 1$ on $[-J, -K]$ and $f(x) = 0$ on $x > -K$. Then for $x > 0$ by (3.1) and (3.8)

$$\begin{aligned} H_n[w_Q^2; f](x) &= \sum_{k=1}^{n} \left(1 - \frac{p_n''(x_{kn})}{p_n'(x_{kn})}(x - x_{kn})\right) f(x_{kn}) l_{kn}^2(x) \\ &= \sum_{x_{kn} \leqslant -K} \left(1 - \frac{p_n''(x_{kn})}{p_n'(x_{kn})}(x - x_{kn})\right) f(x_{kn}) l_{kn}^2(x) \\ &\sim \sum_{x_{kn} \leqslant -K} -Q'(x_{kn})(x - x_{kn}) f(x_{kn}) l_{kn}^2(x). \end{aligned}$$

Let y_n be an element to maximize $|p_n w_Q|$ and for some constant $0 < C_1 < C_2$ and large n, $a_{n/2} < a_n(1 - C_2\delta_n) < y_n < a_n(1 - C_1\delta_n) < a_n$ by (A.3), (A.9), and (A.10). Then we have for large n and for some $0 < \beta < 1$ with $J < \beta a_n$

$$\begin{aligned} x &\left| H_n[w_Q^2; f](y_n) w_Q^2(y_n) v(y_n) \right| \\ &\sim \sum_{x_{kn} \leqslant -K} -Q'(x_{kn})(y_n - x_{kn}) f(x_{kn}) l_{kn}^2(y_n) w_Q^2(y_n) v(y_n) \\ &\gtrsim \sum_{-\beta a_n < x_{kn} < -J} -Q'(x_{kn})(y_n - x_{kn}) f(x_{kn}) l_{kn}^2(y_n) w_Q^2(y_n) v(y_n) \\ &\sim \sum_{-\beta a_n < x_{kn} < -J} (y_n - x_{kn}) l_{kn}^2(y_n) w_Q^2(y_n) v(y_n) w_Q^{-2}(x_{kn}) \log^{-1}(1 + |x_{kn}|). \end{aligned}$$

Since for $-\beta a_n < x_{kn} < -J$ and by (A.1) and (A.2),

$$\begin{aligned} &\Psi_n(x_{kn}) \sim \left(1 - |x_{kn}|/a_n + L\delta_n\right)^{1/2} \sim 1, \\ &\Delta x_{kn} := x_{kn} - x_{k+1,n} \sim \frac{a_n}{n}\Psi_n(x_{kn}) \sim \frac{a_n}{n}, \end{aligned}$$

and by (A.9),

$$a_n^{1/2} p_n(y_n) w_Q(y_n) \sim (nT(a_n))^{1/6},$$

we have for $-\beta a_n < x_{kn} < -J$ by (A.4)

$$\begin{aligned} &(y_n - x_{kn}) l_{kn}^2(y_n) w_Q^{-2}(x_{kn}) w_Q^2(y_n) \\ &\sim (y_n - x_{kn}) \frac{a_n^3}{n^2} \Psi_n^2(x_{kn}) (1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{p_n^2(y_n) w_Q^2(y_n)}{(y_n - x_{kn})^2} \\ &\sim \frac{a_n}{n} (nT(a_n))^{1/3} \frac{\Delta x_{kn}}{y_n + \beta a_n}. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\left| H_n[w_Q^2; f](y_n) w_Q^2(y_n) v(y_n) \right| \\ &\gtrsim \frac{a_n}{n} (nT(a_n))^{1/3} v(y_n) \sum_{-\beta a_n < x_{kn} < -J} \frac{\Delta x_{kn}}{\left(y_n + \beta a_n\right) \log a_n} \end{aligned}$$

$$
\begin{aligned}
&\sim v(y_n)\frac{a_n}{n}\,(nT(a_n))^{1/3}\,\frac{1}{\log a_n}\\
&\sim v(y_n)a_n\frac{T^{1/3}(a_n)}{n^{2/3}}\frac{1}{\log a_n}\\
&\sim v(y_n)a_n Q^{-2/3}(a_n)\frac{1}{\log a_n} \quad \text{by (A.5)}\\
&\sim v(y_n)y_n Q^{-2/3}(y_n)\frac{1}{\log y_n} \to \infty \quad \text{by (A.6)}
\end{aligned}
$$

as $n \to \infty$, since $a_{n/2} < y_n < a_n$. Therefore, we have the result. □

Proof of Theorem 2.6. Let $0 < K$, $K \in I+$ and K be a constant satisfying the conditions of Lemma 3.3. See Eq. (3.8). Define a continuous function g satisfying that for some constant L with $L > K$

$$
\begin{cases}
g(x) = 0, & x > -K,\\
0 \leqslant g(x)w_Q^2(x)u(x) \leqslant L^{-1/p}\log^{-1-1/p}(1+L), & -L \leqslant x \leqslant -K,\\
\text{and} & \\
g(x)w_Q^2(x)u(x) = |x|^{-1/p}\log^{-1-1/p}(1+|x|), & x < -L.
\end{cases}
$$

Then g satisfies (2.7). Let γ be a constant with $0 < \gamma < 1$ and $a_{\gamma n} > L$ for large n. Then for large n, $0 < \gamma < \delta < 1$, and $x > 0$

$$
\begin{aligned}
&\left|H_n[w_Q^2; g](x)\right|\\
&\sim \sum_{x_{kn} < -K} -g(x_{kn})Q'(x_{kn})(x - x_{kn})l_{kn}^2(x) \quad \text{by (3.8)}\\
&\gtrsim \sum_{-a_{\delta n} < x_{kn} < -a_{\gamma n}} -g(x_{kn})Q'(x_{kn})(x - x_{kn})l_{kn}^2(x)\\
&\gtrsim \sum_{-a_{\delta n} < x_{kn} < -a_{\gamma n}} -Q'(x_{kn})(x - x_{kn})l_{kn}^2(x)w_Q^{-2}(x_{kn})u^{-1}(x_{kn})|x_{kn}|^{-1/p}\\
&\quad \times \log^{-1-1/p}(1+|x_{kn}|).
\end{aligned}
$$

Since for $-a_{\delta n} < x_{kn} < -a_{\gamma n}$ by (A.1), (A.7), and (A.6)

$$
\left(1 - |x_{kn}|/a_n + L\delta_n\right)^{1/4} \sim T^{-1/4}(a_n), \quad \Psi_n(x_{kn}) \sim T^{-1/2}(a_n),
$$

$$
T(x_{kn}) \sim T(a_n), \text{ and } -Q'(x_{kn}) \sim Q'(a_n),
$$

by (A.2), (A.4), and (A.5) we have

$$
-Q'(x_{kn})(x - x_{kn})l_{kn}^2(x)w_Q^{-2}(x_{kn}) \sim a_n p_n^2(x)T^{-1/2}(a_n)\frac{\Delta x_{kn}}{x + |x_{kn}|}.
$$

Therefore, for $\sigma > 1$ and $0 < x < a_{\sigma n}$ we obtain

$$\begin{aligned}
&\left|H_n[w_Q^2; g](x)\right| \\
&\quad \gtrsim a_n^{1-1/p} p_n^2(x) T^{-1/2}(a_n) \log^{-1-1/p} a_n \sum_{-a_{\delta n} < x_{kn} < -a_{\gamma n}} \frac{\Delta x_{kn}}{(x + a_{\delta n})} u^{-1}(a_{\delta n}) \\
&\quad \sim a_n^{1-1/p} p_n^2(x) T^{-1/2}(a_n) \log^{-1-1/p} a_n \frac{a_{\delta n} - a_{\gamma n}}{(x + a_{\delta n})} u^{-1}(a_{\delta n}) \\
&\quad \sim a_n^{1-1/p} p_n^2(x) T^{-1/2}(a_n) \log^{-1-1/p} a_n \frac{(1 - a_{\gamma n}/a_{\delta n})}{(1 + x/a_{\delta n})} u^{-1}(a_{\delta n}) \\
&\sim a_n^{1-1/p} p_n^2(x) \frac{T^{-3/2}(a_n)}{1 + x/a_{\delta n}} u^{-1}(a_{\delta n}) \log^{-1-1/p} a_n \quad \text{by (A.7)} \\
&\sim a_n^{1-1/p} p_n^2(x) T^{-3/2}(a_n) u^{-1}(a_{\delta n}) \log^{-1-1/p} a_n.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\limsup_{n\to\infty} u^{-1}(a_{\delta n}) T^{-3/2}(a_n) a_n^{1-1/p} \left(\int_{|x| \leqslant a_{\sigma n}} \left|p_n^2(x) w(x)\right|^p dx\right)^{1/p} / \log^{1+1/p} a_n \\
&\quad \lesssim \limsup_{n\to\infty} \left(\int_I |H_n[w_Q^2; g](x) w(x)|^p dx\right)^{1/p},
\end{aligned}$$

implying the result. □

Proof of Theorems 2.4 and 2.7. By applying the uniform boundedness theorem for $1 \leqslant p \leqslant \infty$, we have that

$$\limsup_{n\to\infty} \left(\int_I |H_n[w_Q^2; g](x) w(x)|^p dx\right)^{1/p} \lesssim \left\|g(x) w_Q^2(x) u(x)\right\|_{L_p(I)} < \infty.$$

For $0 < p < 1$, the method is almost the same as in the case of $1 \leqslant p \leqslant \infty$ (cf. Theorem 10.19 [27, p. 182]). □

4. Applications: $\mathcal{F}$ cases and $\mathcal{E}_1$ cases

We consider Freud weights $w := \exp(-Q)$ where $Q : \mathbb{R} \to \mathbb{R}$ is of polynomial growth at infinity.

Definition 4.1. Freud class $\mathcal{F}$:
Let $w_Q \in \mathcal{A}$ on $I = \mathbb{R}$ satisfying (b1). Then we write $w_Q \in \mathcal{F}$.

A typical example of the Freud weights is

$$w_\alpha(x) := \exp(-|x|^\alpha), \quad \alpha > 1, \quad x \in \mathbb{R}.$$

Specially, the case of $\alpha = 2$ is the Hermite weight.

Proposition 4.2. *Let* $w_Q \in \mathcal{F}$. *If A and B are the same as in* (2.3) *then for some constant* $C > 0$

$$Cu^{1/B} \leqslant a_u \leqslant Cu^{1/A}, \quad u \in [1, \infty). \tag{4.1}$$

Proof. This is Lemma 5.2 (b) [13, p. 478]. □

In the following, we restate Corollary 5.3 of [11].

Proposition 4.3 (*Jung [11]*). *Let* $w_Q \in \mathcal{F}$. *Let* $u(x) := (1 + |Q'(x)|)$ *and* $v(x) := (1 + |x|)^{-\Delta}$, $\Delta \geqslant 0$. *If* $\Delta > 1 - \frac{2}{3}A$ *then for a continuous function f on* $\mathbb{R}$ *with*

$$\lim_{|x|\to\infty} \left| f(x) w_Q^2(x) u(x) \right| = 0,$$

we have

$$\lim_{n\to\infty} \left\| \left(f(x) - H_n[w_Q^2; f](x) \right) w_Q^2(x) v(x) \right\|_{L_\infty(\mathbb{R})} = 0$$

where A is the same as in (2.3).

Theorem 4.4. *Let* $w_Q \in \mathcal{F}$. *Let* $u(x) := (1 + |Q'(x)|)$ *and* $v(x) := (1 + |x|)^{-\Delta}$, $\Delta \geqslant 0$. *If for every continuous function g defined on I satisfying* (2.4) it holds that

$$\lim_{n\to\infty} \left\| (g(x) - H_n[w_Q^2; g](x)) w_Q^2(x) v(x) \right\|_{L_\infty(\mathbb{R})} = 0$$

then it is necessary that

$$\Delta \geqslant \max\left\{0, 1 - \frac{2}{3}B\right\},$$

where B is the same as in (2.3).

Proof. From (2.5), it is necessary that

$$\begin{aligned} &\lim_{|x|\to\infty} x(1+x)^{-\Delta} Q^{-2/3}(x)/\log(1+|x|) < \infty \\ &\quad = \lim_{x\to+\infty} x^{1-\Delta} Q^{-\frac{2}{3}}(x)/\log x. \end{aligned} \tag{4.2}$$

Since from (2.3) it follows that

$$C_1 x^A \leqslant Q(x) \leqslant C_2 x^B, \quad |x| \geqslant x_0 > 0,$$

if $\lim_{x\to+\infty} x^{1-\Delta}(x^B)^{-\frac{2}{3}}/\log x$ diverges, then (4.2) also diverges which implies the statement. □

Proposition 4.5 (*Jung and Kwon [12]*). *Let* $w_Q \in \mathcal{F}$ *and let* $0 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, *and* $\widehat{\alpha} := \min\{1, \alpha\}$. *Assume that for* $0 < p \leqslant 2$,

$$\Delta > -\widehat{\alpha} + \frac{1}{p};$$

and for $p > 2$,

$$a_n^{-(\widehat{\alpha}+\Delta)+\frac{1}{p}} n^{\frac{1}{3}(1-\frac{2}{p})} = O\left(\frac{1}{\log n}\right) \text{ as } n \to \infty,$$

where a_n *is the nth Mhaskar–Rakhmanov–Saff number. Then*

$$\lim_{n\to\infty} \left\|(f(x) - H_n[w_Q^2; f](x))w_Q^2(x)(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})} = 0,$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|\, w_Q^2(x)(1+|x|)^{\alpha} = 0.$$

Theorem 4.6. *Let* $\alpha \geqslant 0$ *and* $0 < p \leqslant \infty$. *If for every continuous function f defined on I satisfying that*

$$\left\|f(x)w_Q^2(x)(1+|x|)^{\alpha}\right\|_{L_p(\mathbb{R})} < \infty,$$

it holds that

$$\lim_{n\to\infty} \left\|(f(x) - H_n[w_Q^2; f](x))w_Q^2(x)(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})} = 0$$

then it is necessary that in case of $0 < p \leqslant 2$,

$$\Delta \geqslant -\alpha,$$

and in case of $2 < p \leqslant \infty$,

$$a_n^{-\alpha-\Delta} n^{\frac{1}{3}\left(1-\frac{2}{p}\right)} = O(\log^{1+1/p} n).$$

Proof. Let $w(x) := w_Q^2(x)(1+|x|)^{-\Delta}$ and $u(x) := (1+|x|)^{\alpha}$. Then by (2.3)

$$\begin{aligned} J &:= u^{-1}(a_{\gamma n})T^{-\frac{3}{2}}(a_n)a_n^{1-1/p} \left\|p_n^2(x)w(x)\right\|_{L_p(|x|\leqslant a_{2n})} \log^{-(1+1/p)} a_n \\ &\sim a_n^{-\alpha-1/p} \left\|a_n p_n^2(x)w_Q^2(x)(1+|x|)^{-\Delta}\right\|_{L_p(|x|\leqslant a_{2n})} \log^{-(1+1/p)} a_n. \end{aligned}$$

From Theorem 2.7, it is necessary that by (A.9)

$$\begin{aligned} \infty > J &\gtrsim a_n^{-\alpha-\Delta-1/p} \left\|a_n p_n^2(x)w_Q^2(x)\right\|_{L_p[-a_n,a_n]} \log^{-(1+1/p)} a_n \\ &\sim a_n^{-\alpha-\Delta-1/p} \left\|a_n p_n^2(x)w_Q^2(x)\right\|_{L_p(\mathbb{R})} \log^{-(1+1/p)} a_n \\ &\sim a_n^{-\alpha-\Delta} \log^{-(1+1/p)} a_n \begin{cases} 1, & p < 2, \\ (\log n)^{1/2}, & p = 2, \\ n^{\frac{1}{3}(1-\frac{2}{p})}, & p > 2. \end{cases} \end{aligned}$$

Therefore, we have the result. □

4.1. Erdős weight case

We consider Erdős weights $w_Q(x) := \exp(-Q(x))$ where $Q : \mathbb{R} \to \mathbb{R}$ is even and is of faster than polynomial growth at infinity.

Definition 4.7. Erdős class $\mathcal{E}_1$:

Let $w_Q \in \mathcal{A}$ on $I = \mathbb{R}$ satisfying (b2). Moreover, if we have for any $\varepsilon > 0$

$$T(x) \leqslant C(Q(x))^{\varepsilon},\ x \to \infty \tag{4.3}$$

for some positive constant C independent of x then we write $w \in \mathcal{E}_1$.

The archetypal examples of $w_Q \in \mathcal{E}_1$ are

(1)

$$w_{k,\alpha}(x) := \exp(-\exp_k(|x|^{\alpha})), \quad x \in \mathbb{R},$$

where $\alpha > 0$, k is a positive integer, and $\exp_k() = \exp(\exp(\exp(\cdots)))$ denotes the kth iterated exponential.

(2)

$$w_{A,B}(x) := \exp\left(-\exp\left(\log(A + x^2)\right)^{B}\right), \quad x \in \mathbb{R},$$

where A is a fixed but large enough positive number and $B > 1$.

For example for $w_{k,\alpha}$,

$$T(x) = T_{k,\alpha}(x) = \alpha\left[1 + x^{\alpha}\sum_{l=1}^{k}\prod_{j=1}^{l-1}\exp_j\left(x^{\alpha}\right)\right], \quad x \geqslant 0$$

and so (2.2) and (4.3) hold in the stronger form

$$\lim_{|x|\to\infty \text{ or } 1} T(x) \Big/ \left(\frac{xQ'(x)}{Q(x)}\right) = 1; \quad \lim_{x\to\infty} \frac{T(x)}{\left[\prod_{j=1}^{k}\log_j Q(x)\right]} = \alpha.$$

Proposition 4.8. *Let $w_Q \in \mathcal{E}_1$. Then for any $\varepsilon > 0$*

$$a_n \leqslant Cn^{\varepsilon} \quad \textit{and} \quad T(a_n) \leqslant Cn^{\varepsilon}, \quad n \geqslant 1. \tag{4.4}$$

Proof. This is Lemma 2.4 in [8]. □

Proposition 4.9 (*Jung [11]*). *Let $w_Q \in \mathcal{E}_1$. Let $u(x) := Q'(x)^{\Delta}$ with $\Delta > 1/3$. Then for a continuous function f on $\mathbb{R}$ with*

$$\lim_{|x|\to\infty}\left|f(x)w_Q^2(x)u(x)\right| = 0,$$

we have

$$\lim_{n\to\infty}\left\|\left(f(x) - H_n[w_Q^2, f](x)\right)w_Q^2(x)\right\|_{L_\infty(\mathbb{R})} = 0.$$

Theorem 4.10. *Let $w_Q \in \mathcal{E}_1$. Let $u(x) := Q'^{\Delta}(x)$, $\Delta \geqslant 0$ and $w(x) := w_Q^2(x)$. If for every continuous function g defined on I satisfying* (2.7) *it holds that*

$$\lim_{n\to\infty} \left\| (g(x) - H_n[w_Q^2; g](x)) w_Q^2(x) \right\|_{L_\infty(\mathbb{R})} = 0$$

then it is necessary that $\Delta \geqslant \frac{1}{3}$.

Proof. Let y_n be an element to maximize $|p_n w_Q|$ and for some constant $0 < C_1 < C_2$ and large n, $a_{n/2} < a_n(1 - C_2\delta_n) < y_n < a_n(1 - C_1\delta_n) < a_n$. Then from (2.6), it is necessary that

$$\begin{aligned}
&\limsup_{n\to\infty} Q'^{-\Delta}(a_{\delta n}) T^{-\frac{3}{2}}(a_n) a_n p_n^2(y_n) w(y_n) / \log a_n < \infty \\
&= \lim_{n\to\infty} \left(\frac{a_n}{n T^{\frac{1}{2}}(a_n)} \right)^{\Delta} T^{-\frac{3}{2}}(a_n)(nT(a_n))^{\frac{1}{3}} / \log a_n \quad \text{by (A.5)} \\
&= \lim_{n\to\infty} n^{\frac{1}{3}-\Delta} a_n^{\Delta} T^{-\frac{\Delta}{2}-\frac{3}{2}}(a_n) / \log a_n.
\end{aligned}$$

Therefore, we have the result because we obtain a contradiction by (4.4) if $\Delta < \frac{1}{3}$. □

Proposition 4.11 (*Damelin et al. [6]*). *Let $w_Q \in \mathcal{E}_1$. Let $0 < p < \infty$, $\Delta \in \mathbb{R}$, and $\kappa > 0$. Assume that*

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{2} - \frac{1}{p}\right)\right\}.$$

Then

$$\lim_{n\to\infty} ||(f(x) - H_n[w_Q^2; f](x)) w_Q^2(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |f(x)| w_Q^2(x) \log^{1+\kappa}(1 + |x|) T^{\frac{1}{2}}(x) = 0.$$

Theorem 4.12. *Let $w_Q \in \mathcal{E}_1$ and $0 < p \leqslant \infty$. If for every continuous function f defined on I satisfying that*

$$\left\| f(x) w_Q^2(x) \log^{1+\kappa}(|x| + 1) T^{1/2}(x) \right\|_{L_p(\mathbb{R})} < \infty,$$

it holds that

$$\lim_{n\to\infty} \left\| (f(x) - H_n[w_Q^2; f](x)) w_Q^2(x)(1 + Q(x))^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0$$

then it is necessary that

$$\Delta \geqslant \max\left\{0, \frac{2}{3}\left(\frac{1}{2} - \frac{1}{p}\right)\right\}. \tag{4.5}$$

Proof. Let $w(x) := w_Q^2(x)(1+|Q(x)|)^{-\Delta}$, $u(x) := \log^{1+\kappa}(1+|x|)T^{1/2}(x)$, and

$$A := u^{-1}(a_{\gamma n})T^{-3/2}(a_n)a_n^{1-1/p}\left\|p_n^2(x)w(x)\right\|_{L_p(\mathbb{R})}\log^{-(1+1/p)}a_n.$$

Similar to Theorem 4.6, from Theorem 2.7, it is necessary that

$$\begin{aligned}\infty > A &\sim (\log a_n)^{-(1+\kappa)}T^{-2}(a_n)a_n^{-1/p}\\ &\quad\times\left\|a_n p_n^2(x)w_Q^2(x)(1+|Q(x)|)^{-\Delta}\right\|_{L_p(|x|\leqslant a_{\sigma n})}\log^{-(1+1/p)}a_n\\ &\gtrsim (\log a_n)^{-(1+\kappa)}T^{-2}(a_n)a_n^{-1/p}Q^{-\Delta}(a_n)\left\|a_n p_n^2(x)w_Q^2(x)\right\|_{L_p[0,a_n]}\\ &\quad\times\log^{-(1+1/p)}a_n\\ &\qquad\text{by the monotonicity of } Q\\ &\sim (\log a_n)^{-(1+\kappa)}T^{-2}(a_n)Q^{-\Delta}(a_n)a_n^{1/2-1/p}\log^{-(1+1/p)}a_n\\ &\quad\times\begin{cases}1, & p<2\\ (\log n)^{1/2} & p=2\\ (nT(a_n))^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}, & p>2\end{cases}\quad\text{by (A.9)}.\end{aligned}$$

Therefore, condition (4.5) is necessary because we obtain a contradiction by (4.4) and (A.5) if $\Delta < \max\left\{0, \frac{2}{3}\left(\frac{1}{2}-\frac{1}{p}\right)\right\}$. □

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Appendix

We let a_u, for $u > 0$, be the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)/\sqrt{1-t^2}\,dt.$$

Then under our assumptions on $w_Q(x)$(see [13–15]), a_u is uniquely defined and is continuous and increasing with *u*. One of its usefulness is the Mhaskar–Saff identity

$$\|Pw_Q\|_{L_\infty(I)} = \|Pw_Q\|_{L_\infty[-a_n,a_n]}$$

valid for every $P \in \mathcal{P}_n,\ n \geqslant 1$.

We define some auxiliary quantities which we will need in the sequel. See [13–15].

Set:

$$\delta_n := (nT(a_n))^{-2/3}, n \geqslant 1,$$

which are useful in describing the behavior of $p_n(w^2, x)$ near x_{1n}. For example, for $w_Q \in \mathcal{A}$

$$|x_{1n}/a_n - 1| \leqslant \frac{L}{2}\delta_n,$$

where L is a positive constant independent of n. $\delta_n \to 0$ as $n \to \infty$, since at least $T(x) \geqslant 1$. We also need the sequence of functions

$$\Psi_n(x) := \begin{cases} \max\left\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n}, \frac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}}\right\}, & |x| \leqslant a_n, \\ \Psi_n(a_n), & |x| \geqslant a_n, \end{cases} \tag{A.1}$$

which are useful in describing the spacing of zeros of $p_n(w^2, x)$ and Christoffel functions.

The convergence of interpolation is closely connected to bounds on orthogonal polynomials and related estimates, which we recall now.

Proposition A.1. *Let $w \in \mathcal{A}$.*

(a) *For $n \geqslant 2$ and $1 \leqslant j \leqslant n-1$,*

$$x_{jn} - x_{j+1,n} \sim \frac{a_n}{n}\Psi_n(x_{jn}). \tag{A.2}$$

(b) *For $n \geqslant 1$,*

$$\sup_{x \in I} |p_n(x)| w_Q(x) |1 - |x|/a_n|^{1/4} \sim a_n^{-1/2}. \tag{A.3}$$

(c) *Uniformly for $n \geqslant 1$, $1 \leqslant j \leqslant n$, and $x \in \mathbb{R}$,*

$$|l_{jn}(x)| \sim \frac{a_n^{3/2}}{n}(\Psi_n w_Q)(x_{jn})(1 - |x_{jn}|/a_n + L\delta_n)^{1/4}\left|\frac{p_n(x)}{x - x_{jn}}\right|. \tag{A.4}$$

(d) *Uniformly for $u \geqslant C$ and $j = 0, 1, 2$,*

$$a_u^j Q^{(j)}(a_u) \sim uT(a_u)^{j-1/2}. \tag{A.5}$$

(e) *Let $0 < \alpha < \beta$. Then uniformly for $u \geqslant C$ and $j = 0, 1, 2$,*

$$T(a_{\alpha u}) \sim T(a_{\beta u}), \quad a_{\alpha u} \sim a_{\beta u}, \quad \text{and} \quad Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}). \tag{A.6}$$

(f) *Given any fixed $r > 1$,*

$$\frac{a_{ru}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{A.7}$$

(g) *There exists a constant ε with $0 < \varepsilon < 2$ such that for $n \geqslant 1$,*

$$T(a_n) \lesssim \left(\frac{n}{a_n}\right)^{2-\varepsilon}. \tag{A.8}$$

(h) *Given* $0 < p \leqslant \infty$, *we have for* $n \geqslant 2$,

$$\left\| p_n w_Q \right\|_{L_p(I)} \sim a_n^{\frac{1}{p}-\frac{1}{2}} \begin{cases} 1, & p < 4, \\ (\log n)^{\frac{1}{4}}, & p = 4, \\ (nT(a_n))^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)}, & p > 4. \end{cases} \tag{A.9}$$

(i) *Let* $0 < p \leqslant \infty$. *Let* $K > 0$. *There exist C and N depending only on K, p,* w_Q *such that* $n \geqslant N$ *and* $P \in \mathcal{P}_n$,

$$\left\| P w_Q \right\|_{L_p(I)} \leqslant C \left\| P w_Q \right\|_{L_p(|x| \leqslant a_n(1-K\delta_n))}. \tag{A.10}$$

Proof.

(a) These follow from Corollary 1.2 (a), (b) in [13], Corollary 1.4 (i), (1.35) in [14], and Corollary 1.3 (a), (b) in [15].

(b) These follow from Corollary 1.4 in [13], Corollary 1.5 (i) in [14], and Corollary 1.4 (a) in [15].

(c) It follows from the formula of l_{jn} and Corollary 1.3 in [13], Corollary 1.5 (iii) in [14], and Corollary 1.4 (b) in [15].

(d)–(e) For (b1) case, these follow from (b1) condition, Lemma 5.1 (c), and Lemma 5.2 (c) in [13]. Otherwise, these follow from part of Lemma 3.2 in [14] and Lemma 2.2 in [15].

(f) It follows from Lemma 5.2 (c) in [13], Lemma 3.2 (v) in [14], and Lemma 2.2 (v) in [15].

(g) For (b1) case, since $T(x)$ is bounded, it follows from Lemma 5.2 (b) in [13]. Otherwise, it follows from Lemma 3.2 (iii) in [14] and Lemma 2.2 (viii) in [15].

(h) It follows from Corollary 1.4 in [13], Theorem 1.8 and Corollary 1.5 (ii) in [14], Corollary 1.4 (a) in [15], Theorem 1.1 in [18], and Theorem 1 in [23].

(i) It follows from Theorem 1.8 in [13], Theorem 1.7 in [14], and Theorem 1.5 in [15]. □

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